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IRREDUCIBLE REPRESENTATIONS OF DEFORMED OSCILLATOR ALGEBRA AND q-SPECIAL FUNCTIONS

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Abstract

Different generators of a deformed oscillator algebra give rise to one-parameter families of q-exponential functions and q-Hermite polynomials related by generating functions. Connections of the Stieltjes and Hamburger classical moment problems with the corresponding resolution of unity for the q-coherent states and with 'coordinate' operators - Jacobi matrices, are also pointed out.

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1. Suggestions to change the canonical commutation relations for improving some properties of quantum field theory have already appeared in the works of the founders of quantum mechanics (see [1, 2] and Refs therein).

In the early fifties, E. P. Wigner [1] posed the following question: "what kind of functions, $f(H)$, appearing in the right-hand side of the commutator $[p, x] = i\hbar f(H)$ are compatible with the given expression of the Hamiltonian, H, and the standard equations of motions"? He found that, in the case of the harmonic oscillator Hamiltonian the usual expression $f(H) = \mathbf{1}$ was not unique. These investigations were continued in [3] where, for a special case, now called "q - root of unity", it was found that, together with the fermionic (two-dimensional) and bosonic (infinite dimensional) cases, there exist cases with dimensions equal to m which are related with the parastatistics not connected to the Green's Ansatz.

Twenty years later, the generalization of the Veneziano amplitude, by substitution of the q-Γ-function instead of the standard Γ-function [4], gave rise in the operator formalism to the q-oscillator commutation relation [5],

$$aa^\dagger - qa^\dagger a = 1, \quad 0 < q < 1. \tag{1}$$

A rejuvenation of the problem of oscillator deformations in the end of the 80's was associated with the growing interest in quantum groups. Such popularity appeared after the works [6, 7], in which a deformation based on the relation $AA^\dagger - q^{1/2}A^\dagger A = q^{-N/2}$, was considered in connection with Schwinger's realization of the quantum algebra $su_q(2)$ [8] (for a q-boson description of the $su_q(1,1)$, see [9]). Further interest in the q-oscillator problem was stimulated by research in the multimode case [10], supersymmetries [11], and relations to the q-analysis [12] (for more details and Refs, see [13]).

Different generators of the deformed oscillator algebra give rise to one-parameter families of q-exponential functions [13, 14, 15, 16], q-Hermite polynomials and other q-special functions [17, 18, 19, 20]. Consideration of the resolution of unity (completeness of the system of q-coherent states) for the q-Bargmann - Fock realization of irreducible representations of deformed oscillator algebra, and the spectral properties [18] of the 'coordinate' operator (which is represented as a Jacobi matrix), pointed out deep connections with the classical Stieltjes and Hamburger moment problems [21].

Recently, the q-oscillator was applied to the study of the phonon spectrum in ^{4}He [22], a specific case of the one-dimensional Schrödinger equation [23], different quantum mechanical models [24], and the trapped atom problem.

2. The deformed oscillator algebra, $\mathcal{A}(q)$, is generated by three elements a, $a^\dagger$, N with defining relations

$$aa^\dagger - qa^\dagger a = 1\,, \quad [N,\, a] = -a, \quad [N,\, a^\dagger] = a^\dagger\,. \tag{2}$$

The generator N is considered as an independent element, and we restrict ourselves to the

case of positive real $q \in (0, \infty)$. The algebra $\mathcal{A}(q)$ has a central element [25],

$$\zeta = q^{-N}([N;q] - a^\dagger a); \quad [N;q] := (1-q^N)/(1-q) \tag{3}$$

(for a more general three-generator algebra $\mathcal{A}(q)$ with $[a, a^\dagger] = F(N)$, see [26, 27]).

In the original papers, the irreducible representation of $\mathcal{A}(q)$ with the vacuum state $|0\rangle$ $(a|0\rangle = 0)$ was considered. The oscillator-type representation space $\mathcal{H}_0$, in the basis of eigenvectors of the operator N, is

$$\mathcal{H}_0 = \{\, |n\rangle; \quad n = 0, 1, 2, ...; \quad a|0\rangle = 0, \quad |n\rangle = ([n;q]!)^{-1/2}(a^\dagger)^n|0\rangle \,\}. \tag{4}$$

Due to the existence of a non-trivial central element, ζ, in addition to $\mathcal{H}_0$, the algebra $\mathcal{A}(q)$ has a set of inequivalent irreducible representations $(0 < q < 1)$ in the spaces $\mathcal{H}_\gamma$ $(\gamma \geq \gamma_c = (1-q)^{-1})$ parameterised by the value of the central element $\zeta = -\gamma$ [25], with the spectrum of N, the set of all integers $\mathbb{Z}$. The matrix $a^\dagger$ in the number operator basis is

$$(a^\dagger)_{n\,k} = c_n \delta_{n\,k+1} \;, \quad a^\dagger|n-1\rangle = c_n|n\rangle \;, \quad (c_n)^2 = \gamma q^n + [n;q] \;. \tag{5}$$

These irreducible representations are connected with different symplectic leaves of Poisson brackets in R^3 , which correspond to the quasiclassical limit of the q-oscillator commutation relation [9].

Considering $\mathcal{A}(q)$ as an associative algebra, any invertible transformation of the generators is admissible; in particular, there are some natural sets of the generators:

$$AA^\dagger - q^{1/2}A^\dagger A = q^{-N/2}\,, \quad [N, A] = -A\,, \quad [N, A^\dagger] = A^\dagger, \tag{6}$$

related to the quantum algebra $sl_q(2)$ via the Schwinger realization [6, 7], and the following set related to the $sl_q(2)$ algebra by a contraction procedure with fixed q [25],

$$[\alpha, \alpha^\dagger] = q^{-N}, \quad [N, \alpha] = -\alpha, \quad [N, \alpha^\dagger] = \alpha^\dagger. \tag{7}$$

The equivalence of these generators is given by the equalities $a = q^{N/2}\alpha = q^{N/4}A$ [9, 13], with an obvious one-parameter generalization, namely,

$$a(\lambda) = q^{-\frac{1}{2}\lambda N}a\,, \quad a^\dagger(\lambda) = a^\dagger q^{-\frac{1}{2}\lambda N}\,. \tag{8}$$

This leads to the commutation relations (still one degree of freedom)

$$a(\lambda)a^\dagger(\lambda) - q^{1-\lambda}a^\dagger(\lambda)a(\lambda) = q^{-\lambda N}\,. \tag{9}$$

Sometimes, these generators and relation (9) are called a two-parameter deformed oscillator [28] : $p \leftrightarrow q^{1-\lambda}$ and $r \leftrightarrow q^{-\lambda}$, $aa^\dagger - pa^\dagger a = r^N$. However, they define the same algebra $\mathcal{A}(q)$ in the case of general $q = p/r$.

One more formal parameter $\nu \in R$ can be added by a shift $N \rightarrow N+\nu$. The corresponding set of $\mathcal{A}(q)$ generators is denoted by $W^{\nu}_{p,r}(q)$ [29]. As a consequence of (9), namely,

$$a(\lambda)(a^{\dagger}(\lambda))^{m} = (pa^{\dagger}(\lambda))^{m}a(\lambda) + (pa^{\dagger}(\lambda))^{m-1}r^{N}[m; \frac{r}{p}] \,, \tag{10}$$

the normalized basis vectors of $\mathcal{H}_0$ in terms of $a^{\dagger}(\lambda)$ are given by

$$|n\rangle = ([n; q, \lambda]!)^{-1/2}(a^{\dagger}(\lambda))^{n}|0\rangle$$

with the factorials defined as

$$[n; q, \lambda]! = \prod_{k=1}^{n}[k; q, \lambda] \,, \quad [m; q, \lambda] = q^{\lambda(1-m)}[m; q] \,. \tag{11}$$

3. In the theory of Lie groups and quantum mechanics, special functions appear as particular matrix elements (overlap coefficients) of appropriate operators in corresponding representations (realizations): examples are exponential functions, as coherent states in the Bargmann-Fock representation of $\mathcal{H}_0$, of the usual boson oscillator $[b, b^{\dagger}] = 1$,

$$\exp(\overline{w}z) = \langle w|z\rangle, \quad |z\rangle = \mathrm{e}^{zb^{\dagger}}|0\rangle, \quad b|z\rangle = z|z\rangle, \tag{12}$$

and Hermite polynomials, as eigenvectors of the operator N, in the coordinate representation,

$$H_n(x) \sim \langle n|x\rangle, \quad (b + b^{\dagger})|x\rangle = 2x|x\rangle.$$

The simple action of the annihilation and creation operators in the coherent state representation leads to the generating function of the Hermite polynomials

$$\omega(z;\, x) = \langle \bar{z}|x\rangle = \exp(2xz - \frac{1}{2}z^{2}) \,. \tag{13}$$

The coherent states of the annihilation operator a of the q-oscillator in $\mathcal{H}_0$ were introduced in [5]:

$$a|z\rangle = z|z\rangle, \qquad |z\rangle = \mathrm{e}_q(za^{\dagger})|0\rangle \,, \tag{14}$$

$$\mathrm{e}_q(x) = \sum_{n=0}^{\infty} \frac{x^{n}}{[n;\, q]\,!} \,. \tag{15}$$

In the q-Bargman-Fock space, related to the q-coherent states, the creation operator $a^{\dagger}$ is the operator of multiplication by $\bar{z}$,

$$\begin{gathered} |f\rangle \rightarrow f(z) = \langle z|f\rangle, \\ \langle z|a^{\dagger}|f\rangle = (a\ |z\rangle)^{\dagger}\,|f\rangle = \bar{z}\,\langle z|f\rangle = \bar{z}\,f(z), \end{gathered} \tag{16}$$

the annihilation operator, a, is a q-difference operator $\mathcal{D}_q$, and the q-coherent state is given by the q-exponent (15),

$$a\, f(z) = \mathcal{D}_q\, f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \qquad \langle z|\zeta\rangle = \mathrm{e}_q(\bar{z}\zeta). \tag{17}$$

The q-exponent above (see (15)) is well known in q-analysis [14]. The scalar product in the q-Bargman-Fock realization of $\mathcal{H}_0$ is given by [5]

$$\langle\phi|f\rangle = \frac{1}{2\pi}\int \overline{\phi(z)}\, f(z)\, \mathrm{d}\mu(z), \tag{18}$$

where the measure is defined by the resolution of unity,

$$\frac{1}{2\pi}\int_0^{1/(1-q)}\!\!\int_0^{2\pi} |z\rangle\langle z|\ (\mathrm{e}_q(q|z|^2))^{-1}\,\mathrm{d}\phi\ \mathrm{d}_q|z|^2 = \sum_{n=0}^{\infty}|n\rangle\langle n| = I\ : \tag{19}$$

this completeness relation was proved in [5] using the product representation of the q-exponent (15), namely,

$$\mathrm{e}_q(x) = \left(\prod_{k=0}^{\infty}(1-(1-q)q^k x)\right)^{-1} = \frac{1}{((1-q)x;q)_\infty}, \tag{20}$$

and the Jackson q-integral, $\int_0^b f(x)\,\mathrm{d}_q x = (1-q)\sum_{m=0}^{\infty} q^m b\, f(q^m b)$ [14].

Following the same pattern, other choices for the generators of the q-oscillator algebra $\mathcal{A}(q)$ give rise to different q-exponential functions [9, 13],

$$\alpha|z\rangle_\alpha = z|z\rangle_\alpha, \quad |z\rangle_\alpha = \mathrm{e}_{1/q}(z\alpha^\dagger)|0\rangle, \tag{21}$$

$$A|z\rangle_A = z|z\rangle_A, \quad |z\rangle_A = \mathrm{E}_q(zA^\dagger)|0\rangle, \tag{22}$$

where the symmetric q-exponent is

$$\mathrm{E}_q(x) = \sum_{m=0}^{\infty}\frac{x^m}{[m]_q\,!}, \qquad [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{23}$$

The one-parameter q-exponential function $\exp(z;q,\lambda)$ is connected with the annihilation operator $a(\lambda)$ (8), (9)

$$a(\lambda)|z;\lambda\rangle = z|z;\lambda\rangle, \quad |z;\lambda\rangle = \exp(za(\lambda)^\dagger;q,\lambda)|0\rangle; \tag{24}$$

$$\exp(z;q,\lambda) = \sum_{m=0}^{\infty} q^{\lambda\, n\,(n-1)/2}\frac{x^n}{[n,q]\,!}. \tag{25}$$

The properties of these q-exponents ($\exp(z;q,\lambda)$) are quite different [15, 16]; for example, for $0<q<1$ and $\lambda<0$, the q-exponent $\exp(z;q,\lambda)$ (25) has zero radius of convergence. It would be interesting to relate different q-exponential functions and their properties with particular physical systems.

The corresponding resolution of unity in the (q,λ)-Bargman-Fock realization of $\mathcal{H}_0$, where the annihilation operator (8) acts as a difference operator $\mathcal{D}_q^{(\lambda)}$ [15],

$$\frac{1}{2\pi}\int_0^\infty\!\!\int_0^{2\pi} |z;\lambda\rangle\langle z;\lambda| \, \mathrm{d}\phi \, \mathrm{d}_q\sigma(|z|^2) = \sum_{n=0}^{\infty} |n\rangle\langle n| = I, \tag{26}$$

results in the classical moment problem (MP) [21] for the measure $\mathrm{d}_q\sigma(|z|^2)$,

$$\int_0^\infty x^n \, \mathrm{d}_q\sigma(|z|^2) = s_n(q;\lambda), \quad s_n(q;\lambda) = [n;q,\lambda]\,!\,. \tag{27}$$

Depending on the behaviour of the moments $s_n(q;\lambda)$ as $n\to\infty$, the MP can be determinate (a unique solution, if any: this is the case of the q-oscillator (2)), or indeterminate (many solutions: these cases are realized for the q-oscillators (6) or (7)). The completeness (the system is overcomplete) was proved for $s_n(q;\lambda)=[n;q]\,!$ [5], $s_n(q;\lambda)=[n]_q\,!$ [30], and $s_n(q;\lambda)=[n;q^{-1}]\,!$ [31]. Complete subsystems of q-coherent states (14) (or (24) for $\lambda=0$) are discussed in [32].

The classical MP refers also to q-Hermite polynomials: the latter are nothing but polynomials of the first kind [21] for a Jacobi matrix $\mathcal{J}$ which is constructed as a "generalized coordinate" from the q-oscillator creation and annihilation operators [17],

$$\mathcal{J}(\lambda) = a(\lambda) + a^\dagger(\lambda), \quad \mathcal{J}(\lambda)\,|x\rangle_\lambda = 2x\,|x\rangle_\lambda, \tag{28}$$

$$|x\rangle_\lambda = \sum_{n=0}^{\infty} H_n(x;q,\lambda)|n\rangle. \tag{29}$$

Due to (28), these q-Hermite polynomials satisfy the following three-term recurrence relation:

$$c_n(\lambda)H_{n-1}(x;q,\lambda) + c_{n+1}(\lambda)H_{n+1}(x;q,\lambda) = x\,H_n(x;q,\lambda). \tag{30}$$

The corresponding generating function can be introduced as in the oscillator case (13), $\omega(z,x;\lambda)=\langle\bar z;\lambda|x\rangle_\lambda$, however its form will depend on the chosen generators of $\mathcal{A}(q)$ [17]

$$\langle\bar z;\lambda|(a(\lambda)+a^\dagger(\lambda))|x\rangle_\lambda = (\mathcal{D}_q^{(\lambda)}+z)\,\omega(z,x;\lambda) = 2x\,\omega(z,x;\lambda)\,.$$

This difference equation for $\omega(z,x;\lambda)$ will include two points for $\lambda=0$, 1, so its solution will be given by the 'standard' q-exponent (15) and three points: z, $q^{-\lambda}z$, $q^{1-\lambda}z$ for the general λ. The measure entering into the q-Hermite polynomials $H_n(x;q,\lambda)$ orthogonality relations

is connected with the solution of the Hamburger MP: this measure is known explicitly for some cases (see e.g.[18]). This connection of the MP with Jacobi matrices gives rise to a generalized deformation of the oscillator identifying the matrix $c_k \delta_{n+1,k}\,, c_k > 0$ with an annihilation operator a. Then one gets the Wigner commutation relation $[a,\, a^\dagger] = F(N)$ with $F(n) = c_{n+1}^2 - c_n^2$ and its central element $\zeta = (c^2(N) - a^\dagger a) + const$ (see also [2, 26, 27]). The q-special functions related to the other irreducible representations $\mathcal{H}_\gamma$ of $\mathcal{A}(q)$ are discussed in [31]. In particular, for the generators (2) the normalized q-coherent states exist in $\mathcal{H}_\gamma$ for the creation operator $a^\dagger$ and $z > \gamma_c = (1-q)^{-1}$.

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